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Heat Kernel Asymptotics of Gilkey–Smith Boundary-Value Problem

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The formulation of gauge theories on compact Riemannian manifolds with boundary leads to partial differential operators with Gilkey–Smith boundary conditions, whose peculiar property is the occurrence of both normal and tangential derivatives on the boundary. Unlike the standard Dirichlet or Neumann boundary conditions, this boundary-value problem is not automatically elliptic but becomes elliptic under certain conditions on the boundary operator. We study the Gilkey–Smith boundary-value problem for Laplace-type operators and find a simple criterion of ellipticity. The first non-trivial coefficient of the asymptotic expansion of the trace of the heat kernel is computed and the local leading asymptotics of the heat-kernel diagonal is also obtained. It is shown that, in the non-elliptic case, the heat-kernel diagonal is non-integrable near the boundary, which reflects the fact that the heat kernel is not of trace class. We apply this analysis to general linear bosonic gauge theories and find an explicit condition of ellipticity.

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1 Introduction

Elliptic differential operators on manifolds have proved to play a crucial role in mathematical physics. In particular, the main objects of interest in quantum field theory and statistical physics, such as the effective action and the partition function, are described by the functional determinants, or, which is equivalent, by the zeta-function and the heat kernel of self-adjoint elliptic differential operators. Of particular importance are, of course, the operators of Laplace type or Dirac type.[1] In the case of manifolds with boundary, one has to impose some boundary conditions in order to make a (formally self-adjoint) differential operator self-adjoint and elliptic. Indeed, the boundary conditions are additional ingredients in the theory which have not been fixed *a priori*, and the choice of boundary conditions is, by no means, unique. There are many admissible boundary conditions that guarantee the self-adjointness and ellipticity of the problem. The simplest boundary conditions are the classical Dirichlet and the Neumann ones. In the Dirichlet case one sets to zero at the boundary the value of the field, whereas in the Neumann case the normal derivative of the field is set to zero at the boundary. There exist also slight modifications of the Neumann boundary conditions (called Robin boundary conditions in physical literature) when the normal derivative of the field at the boundary is not set to zero but is proportional to the value of the field at the boundary. An even more general scheme, called mixed boundary conditions, applies to the operators acting on sections of some vector bundles. It is then possible to mix the Dirichlet and Robin boundary conditions by using some projectors, i.e. a part of the field components satisfy Dirichlet boundary conditions and the remaining part satisfies Robin boundary conditions.

However, this is not the most general scheme. A much more general setup for the boundary-value problem was developed by Gilkey and Smith.[2] They put forward some boundary conditions that are still local but include both normal and *tangential derivatives* of the fields at the boundary. In this paper we are going to study the Gilkey–Smith boundary-value problem for operators of Laplace type. Unlike the Dirichlet or Neumann boundary-value problems, such a boundary-value problem is not automatically elliptic. Therefore, following Refs. 1,3, we find, first, a criterion of (strong) ellipticity, which provides an explicit simple condition on the boundary operator. Then we construct the parametrix to the heat equation in the leading approximation and compute the first non-trivial (next to leading) term $A_{1/2}$ in the asymptotic expansion of the trace of the heat kernel. We also discuss what happens when the boundary-value problem is not strongly elliptic. Last, we study the problem of ellipticity in linearized gauge theories on manifolds with boundary. The attempt to preserve gauge invariance on manifolds with boundary fixes the boundary conditions and leads exactly to a Gilkey–Smith boundary-value problem. As is shown in Ref. 1, while Yang–Mills as well as Rarita–Schwinger theories are automatically elliptic, quantum gravity based on the Einstein action turns out to be not elliptic, if the Gilkey–Smith boundary-value problem is studied.

2 Gilkey–Smith Boundary-Value Problem

Let (M, g) be a smooth compact Riemannian manifold of dimension m with smooth boundary ∂M. Let g be the positive-definite Riemannian metric on M and $\hat{g}$ be the induced metric on ∂M. By using the inward geodesic flow, we identify a narrow neighbourhood of the boundary ∂M with a part of $\partial M \times \mathbf{R}_+$ and define a split of the cotangent bundle $T^*(M) = T^*(\partial M) \oplus T^*(\mathbf{R})$. Let $\hat{x} = (\hat{x}^i)$, with $i = 1, 2, \ldots, m-1$, be the local coordinates on ∂M and r be the normal geodesic distance to the boundary, so that $N = \partial_r = \partial/\partial r$ is the inward-pointing unit normal vector field to the boundary. Near ∂M we choose the local coordinates $x = (x^\mu) = (\hat{x}, r)$, with $\mu = 1, 2, \ldots, m$, and the split $\xi = (\xi_\mu) = (\zeta, \omega) \in T^*(M)$, where $\zeta = (\zeta_j) \in T^*(\partial M)$ and $\omega \in \mathbf{R}$. With our notation, Greek indices range from 1 through m and lower case Latin indices range from 1 through $m - 1$.

Let V be a (smooth) vector bundle over the manifold M and $C^\infty(V, M)$ be the space of smooth sections of the bundle V. Let V^* be the dual vector bundle and $E: V \to V^*$ be a Hermitian non-degenerate metric, $E^\dagger = E$, that determines the Hermitian fibre scalar product in V. Using the invariant Riemannian volume element $\mathrm{dvol}(x)$ on M one defines a natural L^2 inner product $(,)$ in $C^\infty(V, M)$, and the Hilbert space $L^2(V, M)$ as the completion of $C^\infty(V, M)$ in this norm.

Let $\nabla^{T^*(M)}$ be the Levi-Civita connection on M and ∇^V be the connection on the vector bundle V compatible with the metric E. Then we define, as usual, $\nabla^{T^*(M)\otimes V} = \nabla^{T^*(M)} \otimes 1 + 1 \otimes \nabla^V$. Moreover, we will often denote just by ∇ the total covariant derivative without mentioning the bundle it is acting on. The notation $\hat{\nabla}$ will be used to denote the covariant tangential derivative on the boundary.

Let, further, $\mathrm{tr}_g = g \otimes 1$ be the contraction of sections of the bundle $T^*(M) \otimes T^*(M) \otimes V$ with the metric on the cotangent bundle $T^*(M)$, and $Q \in C^\infty(\mathrm{End}\,(V), M)$ be a smooth self-adjoint endomorphism of the bundle V, i.e. $\bar{Q} \equiv E^{-1} Q^\dagger E = Q$. Then a Laplace-type operator $F: C^\infty(V, M) \to C^\infty(V, M)$ is a second-order differential operator defined by

$$F \equiv -\mathrm{tr}_g \nabla^{T^*(M)\otimes V} \nabla^V + Q. \tag{2.1}$$

Let us define the *boundary data* by

$$\psi(\varphi) = \begin{pmatrix} \psi_0(\varphi) \\ \psi_1(\varphi) \end{pmatrix}, \tag{2.2}$$

where $\psi_0(\varphi) \equiv \varphi|_{\partial M}$ and $\psi_1(\varphi) \equiv \nabla_N \varphi|_{\partial M}$ are the restrictions to the boundary of the sections $\varphi \in C^\infty(V, M)$ and their normal derivatives. Let the vector bundle W over ∂M be the bundle of the boundary data. W consists of two copies of the restriction of V to ∂M and inherits a natural grading[4] $W = W_0 \oplus W_1$, where W_j represents normal derivatives of order j, and, therefore, $\dim W = 2 \dim V$. The bundles W_0 and W_1 have the same structure, and hence in the following they will be often identified. Let $W' = W_0' \oplus W_1'$ be an auxiliary graded vector bundle over ∂M such that $\dim W' = \dim V$.

Let $B : C^\infty(W, \partial M) \to C^\infty(W', \partial M)$ be a tangential differential operator on ∂M. The boundary conditions then read

$$B\psi(\varphi) = 0. \tag{2.3}$$

Now let Π be a self-adjoint projector acting on W_0. Since the bundle W_1 is identified with W_0 the projector Π acts also on W_1. Let $\Gamma \in C^\infty(T(\partial M) \otimes \mathrm{End}\,(W_0), \partial M)$ be an anti-self-adjoint endomorphism-valued vector field on ∂M orthogonal to Π, i.e. $\bar{\Gamma}^i = -\Gamma^i$, $\Pi\Gamma^i = \Gamma^i\Pi = 0$, and $S \in C^\infty(\mathrm{End}\,(W_0), \partial M)$ be a self-adjoint endomorphism orthogonal to Π, i.e. $\bar{S} = S$, $\Pi S = S\Pi = 0$. Using these objects we define a first-order self-adjoint tangential differential operator $\Lambda :\ C^\infty(W_0, \partial M) \to C^\infty(W_0, \partial M)$ by

$$\Lambda = (I - \Pi)\left\{\frac{1}{2}(\Gamma^i\hat{\nabla}_i + \hat{\nabla}_i\Gamma^i) + S\right\}(I - \Pi), \tag{2.4}$$

that is obviously orthogonal to Π: $\Pi\Lambda = \Lambda\Pi = 0$. Hereafter I is the identity endomorphism of the vector bundle V. The Gilkey–Smith boundary operator is expressed in terms of these geometric objects by

$$B = \begin{pmatrix} \Pi & 0 \\ \Lambda & I - \Pi \end{pmatrix}. \tag{2.5}$$

It is not difficult to see that the Gilkey–Smith boundary-value problem incorporates all standard types of boundary conditions. Indeed, by choosing $\Pi = I$ and $\Lambda = \Gamma = S = 0$ one gets the standard Dirichlet boundary conditions, by choosing $\Pi = 0$, $\Gamma = 0$, $\Lambda = S = I$ one gets the standard Neumann boundary conditions. More generally, the choice $\Gamma = 0$ and $\Lambda = S$ corresponds to the mixed boundary conditions mentioned in Sec. 1.

3 Strong Ellipticity

Integration by parts shows[1] that the Laplace-type operator F given in Eq. (2.1) endowed with the Gilkey–Smith boundary conditions (2.3) is symmetric, meaning that $(\varphi_1, F\varphi_2) = (F\varphi_1, \varphi_2)$ for any two smooth sections $\varphi_1, \varphi_2 \in C^\infty(V, M)$ satisfying the boundary conditions $B\psi(\varphi_1) = B\psi(\varphi_2) = 0$. However, it is not automatically elliptic. Now we are going to determine under which conditions the Gilkey–Smith boundary-value problem for a Laplace-type operator is strongly elliptic.[4]

First of all, the leading symbol of the operator F should be elliptic in the interior of M. Let, hereafter, λ be a complex number which does not lie on the positive real axis, $\lambda \in \mathbf{C} - \mathbf{R}_+$ ($\mathbf{R}_+$ being the set of positive numbers). Then by using the leading symbol of the operator F, i.e. $\sigma_L(F; x, \xi) = |\xi|^2 \cdot I$, with $|\xi|^2 \equiv g^{\mu\nu}(x)\xi_\mu\xi_\nu$, we find easily

$$\det\left(\sigma_L(F; x, \xi) - \lambda \cdot I\right) = (|\xi|^2 - \lambda)^{\dim V} \neq 0 \qquad \text{for } (\xi, \lambda) \neq (0, 0). \tag{3.6}$$

Thus, the leading symbol of the operator F is *elliptic*.

Second, the so-called *strong ellipticity condition* should be satisfied.[2, 4] As we already noted above, there is a natural grading in the vector bundles W and W' which reflects

simply the number of normal derivatives of a section of the bundle.[4] The boundary operator B in Eq. (2.5) is said to have the *graded order* 0. Its *graded leading symbol* is defined by[2, 4]

$$\sigma_g(B_F) \equiv \begin{pmatrix} \Pi & 0 \\ i\Gamma\cdot\zeta & (I-\Pi) \end{pmatrix}, \tag{3.7}$$

where $\Gamma\cdot\zeta \equiv \Gamma^j\zeta_j$. To define the strong ellipticity condition we take the leading symbol $\sigma_L(F;\hat{x},r,\zeta,\omega)$ of the operator F, substitute $r=0$ and $\omega\to -i\partial_r$ and consider the following ordinary differential equation for a section $\varphi\in C^\infty(V,\partial M\times \mathbf{R}_+)$:

$$\left[\sigma_L(F;\hat{x},0,\zeta,-i\partial_r)-\lambda\cdot I\right]\varphi(r)=0, \tag{3.8}$$

with an asymptotic condition

$$\lim_{r\to\infty}\varphi(r)=0. \tag{3.9}$$

The boundary-value problem (F,B) is said to be *strongly elliptic*[2, 4] with respect to the cone $\mathbf{C}-\mathbf{R}_+$ if for every pair $(\zeta,\lambda)\neq(0,0)$, and any $\psi'\in C^\infty(W',\partial M)$ there is a *unique* solution φ of the equation (3.8) satisfying the asymptotic condition (3.9) and the boundary condition

$$\sigma_g(B_F)(\hat{x},\zeta)\psi(\varphi)=\psi', \tag{3.10}$$

where $\psi(\varphi)\in C^\infty(W,\partial M)$ are the boundary data defined by (2.2).

For a Laplace-type operator this definition leads to the following theorem.[1]

Theorem 1 *The Gilkey–Smith boundary-value problem (F,B) is strongly elliptic with respect to $\mathbf{C}-\mathbf{R}_+$ if and only if the matrix $|\zeta|I-i\Gamma\cdot\zeta$ is positive-definite, i.e. $|\zeta|I-i\Gamma\cdot\zeta>0$, for any non-vanishing ζ. A sufficient condition for strong ellipticity is:*

$$|\zeta|^2 I+(\Gamma\cdot\zeta)^2>0. \tag{3.11}$$

4 Asymptotic Expansion of the Heat Kernel

For $t>0$ the heat semi-group $\exp(-tF):\ L^2(V,M)\to L^2(V,M)$ of the strongly elliptic boundary-value problem (F,B) is well defined. The kernel of this operator, $U(t|x,y)$, called heat kernel, is a section of the tensor product of the vector bundles V and V^* over the tensor-product manifold $M\times M$, defined by the equation

$$(\partial_t+F)U(t|x,y)=0 \tag{4.12}$$

with initial condition

$$U(0^+|x,y)=\delta(x,y), \tag{4.13}$$

where $\delta(x,y)$ is the covariant Dirac distribution. Moreover, a boundary condition is imposed, i.e.

$$B\psi[U(t|x,y)]=0, \tag{4.14}$$

and the symmetry condition holds

$$U(t|x,y) = U(t|y,x). \tag{4.15}$$

Hereafter all differential operators as well as the boundary data map act on the *first* argument of the heat kernel, unless otherwise stated.

It is well known[4] that the heat kernel $U(t|x,y)$ is a smooth function near diagonal of $M \times M$ and has a well defined diagonal value $U(t|x,x)$, and that the L^2 trace

$$\mathrm{Tr}\,_{L^2} \exp(-tF) = \int_M \mathrm{dvol}(x) \mathrm{tr}\,_V U(t|x,x), \tag{4.16}$$

has an asymptotic expansion as $t \to 0^+$

$$\mathrm{Tr}\,_{L^2} \exp(-tF) \sim (4\pi t)^{-m/2} \sum_{k \geq 0} t^{k/2} A_{k/2}(F,B). \tag{4.17}$$

Here $A_{k/2}(F,B)$ are the famous *global* heat-kernel coefficients (sometimes called also Minakshisundaram–Plejel or Seeley coefficients). The zeroth-order coefficient is very well known:

$$A_0 = \int\limits_M \mathrm{dvol}(x) \mathrm{tr}\,_V I = \mathrm{vol}\,(M) \cdot \dim(V). \tag{4.18}$$

It is independent of the operator F and of the boundary conditions B. The higher order coefficients have the following general form:

$$A_{k/2}(F,B) = \int\limits_M \mathrm{dvol}(x) \mathrm{tr}\,_V a_{k/2}(F|x) + \int\limits_{\partial M} \mathrm{dvol}(\hat{x}) \mathrm{tr}\,_V b_{k/2}(F,B|\hat{x}), \tag{4.19}$$

where $a_{k/2}(F|x)$ and $b_{k/2}(F,B|\hat{x})$ are the (*local*) *interior* and *boundary* heat-kernel coefficients. The interior coefficients $a_{k/2}(F|x)$ do *not* depend on the boundary conditions. Moreover, it is well known that they vanish for half-integer order, $a_{k+1/2} = 0$.[4] The integer order coefficients $a_k(F|x)$ are calculated for Laplace-type operators up to a_4.[5] The boundary coefficients $b_{k/2}(F,B|\hat{x})$ do depend on *both* the operator F and the boundary operator B. They are far more complicated because in addition to the geometry of the manifold M they depend essentially on the geometry of the boundary ∂M. For Laplace-type operators they are known for the usual boundary conditions (Dirichlet, Neumann, or mixed version of them) up to $b_{5/2}$.[6, 7] For Gilkey–Smith boundary conditions only some special cases have been studied in the literature.[8, 9, 10, 11, 12] In this paper we evaluate the next-to-leading coefficient $A_{1/2}(F,B)$, following our recent work.[1]

5 Parametrix: General Setup

In this section we show how one can construct an approximation to the heat kernel $U(t|x,y)$ for $t \to 0^+$ near the diagonal, i.e. for x close to y. First of all, we decompose

the heat kernel into two parts

$$U(t|x,y) = U_\infty(t|x,y) + U_B(t|x,y). \tag{5.20}$$

Then we construct *different* approximations for U_∞ and U_B. The first part $U_\infty(t|x,y)$ is approximated by the usual asymptotic expansion of the heat kernel in the case of compact manifolds *without boundary* when $x \to y$ and $t \to 0^+$. This means that effectively one introduces a small expansion parameter ε reflecting the fact that the points x and y are close to each other and the parameter t is small. This can be done by fixing a point x', choosing the normal coordinates at this point (with $g_{\mu\nu}(x') = \delta_{\mu\nu}$) and scaling

$$x \to x' + \varepsilon(x - x'), \qquad y \to x' + \varepsilon(y - x'), \qquad t \to \varepsilon^2 t, \tag{5.21}$$

and expanding into an asymptotic series in ε. This construction is, by now, quite standard[4] and we do not repeat it here. One can also use a completely covariant method,[5] which leads to the result

$$U_\infty(t|x,y) \sim (4\pi t)^{-m/2} \exp\left(-\frac{d^2(x,y)}{4t}\right) \sum_{k\geq 0} t^k a_k(x,y), \tag{5.22}$$

where $d(x,y)$ is the geodesic distance between x and y and $a_k(x,y)$ are the off-diagonal heat-kernel coefficients. These coefficients satisfy certain differential recursion relations which can be solved in form of a covariant Taylor series near diagonal.[5] On the diagonal the asymptotic expansion of the heat kernel reads

$$U_\infty(t|x,x) \sim (4\pi t)^{-m/2} \sum_{k\geq 0} t^k a_k(F|x), \tag{5.23}$$

where $a_k(F|x) \equiv a_k(x,x)$. As we noted above, the explicit formulae for the diagonal values of a_k are known up to $k = 4$.[5] This asymptotic expansion can be integrated over the manifold M to get

$$\int\limits_M \mathrm{dvol}(x) \mathrm{tr}\,_V U_\infty(t|x,x) \sim (4\pi t)^{-m/2} \sum_{k\geq 0} t^k \int\limits_M \mathrm{dvol}(x) \mathrm{tr}\,_V\, a_k(F|x). \tag{5.24}$$

Thus, integrating the diagonal of U_∞ gives the interior terms in the heat-kernel asymptotics (4.19).

For a *strongly elliptic* boundary-value problem the diagonal of the boundary part $U_B(t|x,x)$ is *exponentially small* as $t \to 0^+$ if $x \notin \partial M$, i.e. of order $\sim \exp(-r^2(x)/t)$, where $r(x)$ is the normal geodesic distance from x to the boundary. Thus, it does not contribute to the asymptotic expansion of the heat-kernel diagonal outside the boundary as $t \to 0^+$. This implies that the asymptotic expansion of the total heat-kernel diagonal outside the boundary is determined only by U_∞

$$U(t|x,x) \sim (4\pi t)^{-m/2} \sum_{k\geq 0} t^k a_k(F|x), \qquad x \notin \partial M. \tag{5.25}$$

The coefficients of the asymptotic expansion as $t \to 0^+$ of the diagonal of the boundary part $U_B(t|x,x)$ behave near the boundary like the one-dimensional Dirac distribution $\delta(r(x))$ and its derivatives. Thus, the *integral* over the manifold M of the boundary part $U_B(t|x,x)$ has an asymptotic expansion as $t \to 0^+$ with non-vanishing coefficients in form of integrals over the boundary. The local boundary coefficients $b_{k/2}$ contribute, after integration over the boundary, to the global heat-kernel coefficients, according to Eq. (4.19). It is well known that the coefficient $A_{1/2}$ is a purely boundary contribution.[4] It is almost obvious that it can be evaluated by integrating the fibre trace of the boundary contribution U_B of the heat kernel to leading order.

Of course, U_∞ is obtained *without* taking into account the boundary conditions. Therefore, it satisfies approximately the equation (4.12) but does *not* satisfy the boundary conditions (4.14). This implies that the compensating term $U_B(t|x,y)$ should be defined by the equation

$$(\partial_t + F)U_B(t|x,y) = 0 \tag{5.26}$$

with the initial condition

$$U_B(0^+|x,y) = 0, \tag{5.27}$$

and the boundary condition

$$B\psi\left[U_\infty(t|x,y) + U_B(t|x,y)\right] = 0. \tag{5.28}$$

The compensating term $U_B(t|x,y)$ is important only near the boundary where it behaves like a distribution when $t \to 0^+$. Since the points x and y are close to the boundary the coordinates $r(x)$ and $r(y)$ are small *separately*, hence not only the difference $[r(x) - r(y)]$ but also the sum $[r(x) + r(y)]$ is small. This means that we must additionally scale $r(x) \to \varepsilon r(x)$ and $r(y) \to \varepsilon r(y)$. By contrast, the point $\hat{x}'$ is kept fixed on the boundary, so the coordinates $\hat{x}'$ do not scale at all: $\hat{x}' \to \hat{x}'$.

Thus, we shall scale the coordinates $x = (\hat{x}, r(x))$, $y = (\hat{y}, r(y))$, and the parameter t according to

$$\hat{x} \to \hat{x}' + \varepsilon(\hat{x} - \hat{x}'), \qquad \hat{y} \to \hat{x}' + \varepsilon(\hat{y} - \hat{x}'), \tag{5.29}$$

$$r(x) \to \varepsilon r(x), \qquad r(y) \to \varepsilon r(y), \qquad t \to \varepsilon^2 t. \tag{5.30}$$

The corresponding differential operators are scaled by

$$\hat{\partial} \to \frac{1}{\varepsilon}\hat{\partial}, \qquad \partial_r \to \frac{1}{\varepsilon}\partial_r, \qquad \partial_t \to \frac{1}{\varepsilon^2}\partial_t. \tag{5.31}$$

We call this transformation just *scaling* and denote the scaled objects by an index ε, e.g. U_B^ε. The scaling parameter ε is considered as a small parameter in the theory and we use it to expand everything in power series in ε. We do *not* take care about the convergence properties of these expansions and take them as *formal* power series. In fact, they are asymptotic expansions as $\varepsilon \to 0$. At the very end of calculations we can set $\varepsilon = 1$. The non-scaled objects, i.e. those with $\varepsilon = 1$, will not have the index ε, e.g.

$U^\varepsilon_B|_{\varepsilon=1} = U_B$. Another way of doing this is by saying that we expand all quantities in the Taylor series in the boundary coordinates $\hat{x}$ and $\hat{y}$ about the point $\hat{x}'$ with the coefficients being homogeneous functions of $r(x)$, $r(y)$ and t.

First of all, we expand the scaled operator F^ε in power series in ε

$$F^\varepsilon \sim \sum_{n\geq 0} \varepsilon^{n-2} F_n, \tag{5.32}$$

where F_n are second-order differential operators with homogeneous symbols. The boundary operator requires a more careful handling. Since half of the boundary data (2.2) contain normal derivatives, formally $\psi_0 = \varphi|_{r=0}$ and $\psi_1 = \partial_r \varphi|_{r=0}$ would be of different order in ε. To make them of the same order we have to assume an additional factor ε in all $\psi_1 \in C^\infty(W_1, \partial M)$. Thus, we define the *graded scaling* of the boundary data map by

$$\psi^\varepsilon(\varphi) = \begin{pmatrix} \psi^\varepsilon_0(\varphi) \\ \varepsilon\psi^\varepsilon_1(\varphi) \end{pmatrix} = \begin{pmatrix} \varphi(\hat{x}, r)|_{r=0} \\ \partial_r\varphi(\hat{x}, r)|_{r=0} \end{pmatrix} = \psi(\varphi), \tag{5.33}$$

so that the boundary data map ψ *does not scale* at all. This leads to an additional factor ε in the operator Λ determining the boundary operator B of Eq. (2.5). Thus, we define the *graded scaling* of the boundary operator by

$$B^\varepsilon = \begin{pmatrix} \Pi^\varepsilon & 0 \\ \varepsilon\Lambda^\varepsilon & I - \Pi^\varepsilon \end{pmatrix}, \tag{5.34}$$

which has the following asymptotic expansion in ε:

$$B^\varepsilon \sim \sum_{n\geq 0} \varepsilon^n B_{(n)}, \tag{5.35}$$

where $B_{(n)}$ are first-order tangential operators with homogeneous symbols. At zeroth order we have

$$F_0 = -\partial_r^2 - \hat{\partial}^2, \tag{5.36}$$

$$B_{(0)} = \begin{pmatrix} \Pi_0 & 0 \\ \Lambda_0 & I - \Pi_0 \end{pmatrix}, \tag{5.37}$$

where

$$\hat{\partial}^2 = \hat{g}^{jk}(\hat{x}')\hat{\partial}_j\hat{\partial}_k, \qquad \Lambda_0 = \Gamma^j(\hat{x}')\hat{\partial}_j, \qquad \Pi_0 = \Pi(\hat{x}'). \tag{5.38}$$

Note that all leading-order operators F_0, $B_{(0)}$ and Λ_0 have *constant* coefficients and, therefore, are very easy to handle. This procedure is called sometimes "*freezing the coefficients* of the differential operator".

The subsequent strategy is rather simple. Expand the scaled heat kernel in ε and substitute into the scaled version of the equation (5.26) and of the boundary condition (5.28). Then, by equating the terms of the same order in ε one gets an infinite set of recursive equations which determine all $U_{B(n)}$. The $U_{\infty(n)}$ are obtained simply by expanding the scaled version of (5.22) in power series in ε.

6 Parametrix: Leading Order

In this section we determine the parametrix of the heat equation to leading order, i.e. $U_0 = U_{\infty(0)} + U_{B(0)}$. As we already outlined above, we fix a point $\hat{x}' \in \partial M$ on the boundary and the normal coordinates at this point (with $\hat{g}_{ik}(\hat{x}') = \delta_{ik}$), take the tangent space $T(\partial M)$ and replace the manifold M by $M_0 \equiv T(\partial M) \times \mathbf{R}_+$. By using the explicit form of the zeroth-order operators F_0, B_0 and Λ_0 given by (5.36)–(5.38) we obtain the equation

$$\left(\partial_t - \partial_r^2 - \hat{\partial}^2\right) U_0(t|x,y) = 0, \tag{6.39}$$

and the boundary conditions

$$\Pi_0 U_0(t|x,y)\Big|_{r(x)=0} = 0, \tag{6.40}$$

$$(I - \Pi_0)\left(\partial_r + i\Gamma_0^j \hat{\partial}_j\right) U_0(t|x,y)\Big|_{r(x)=0} = 0, \tag{6.41}$$

where $\Pi_0 = \Pi(\hat{x}'), \Gamma_0^j = \Gamma^j(\hat{x}')$. Hereafter the differential operators always act on the first argument of a kernel. Moreover, for simplicity of notation, we will denote Π_0 and Γ_0 just by Π and Γ^j and omit the dependence of all geometric objects on $\hat{x}'$. To leading order this cannot cause any misunderstanding. Furthermore, the heat kernel should be bounded,

$$\lim_{r(x)\to\infty} U_0(t|x,y) = \lim_{r(y)\to\infty} U_0(t|x,y) = 0, \tag{6.42}$$

and symmetric,

$$U_0(t|x,y) = U_0(t|y,x). \tag{6.43}$$

To solve the above boundary-value problem we use the Laplace tarnsform in t and, since it has constant coefficients, the Fourier transform in $(\hat{x} - \hat{y})$. Therefore, it reduces to an *ordinary* differential equation of second order in r on $\mathbf{R}_+$, which can be easily solved taking into account the boundary conditions at $r = 0$ and $r \to \infty$. Omitting simple but lengthy calculations we obtain

$$U_0(t|x,y) = \int\limits_{\mathbf{R}^{m-1}} \frac{d\zeta}{(2\pi)^{m-1}} \int\limits_{w-i\infty}^{w+i\infty} \frac{d\lambda}{2\pi i}\, e^{-t\lambda + i\zeta\cdot(\hat{x}-\hat{y})} G(\lambda|\zeta, r(x), r(y)), \tag{6.44}$$

where w is a negative constant and G is the leading-order resolvent kernel in momentum representation. It reads

$$G(\lambda|\zeta,u,v) = \frac{1}{2\sqrt{|\zeta|^2-\lambda}}\left\{ \exp\left\{-|u-v|\sqrt{|\zeta|^2-\lambda}\right\} \right.$$
$$\left. + \left[I - 2\Pi + 2i\Gamma\cdot\zeta\left(I\sqrt{|\zeta|^2-\lambda} - i\Gamma\cdot\zeta\right)^{-1}\right] \exp\left[-(u+v)\sqrt{|\zeta|^2-\lambda}\right] \right\}, \tag{6.45}$$

where $\mathrm{Re}\,\sqrt{|\zeta|^2-\lambda}>0$. Now, by scaling the integration variables $\lambda\to\lambda/t$ and $\zeta\to\zeta/\sqrt{t}$ and shifting the contour of integration over λ ($w\to w/t$, which can be done because the integrand is analytic in the left half-plane of λ) and using the homogeneity property of the resolvent kernel we obtain immediately

$$\begin{aligned} U_0(t|x,y) &= (4\pi t)^{-m/2}\int\limits_{\mathbf{R}^{m-1}}\frac{d\zeta}{\pi^{(m-1)/2}}\exp\left\{i\zeta\cdot\frac{(\hat{x}-\hat{y})}{\sqrt{t}}\right\} \\ &\quad\times\int\limits_{w-i\infty}^{w+i\infty}\frac{d\lambda}{i\sqrt{\pi}}\,e^{-\lambda}\,G\left(\lambda\Big|\zeta,\frac{r(x)}{\sqrt{t}},\frac{r(y)}{\sqrt{t}}\right). \end{aligned} \tag{6.46}$$

Next, let us change the variable λ according to $\lambda\equiv|\zeta|^2+\omega^2$. In the upper half-plane, $\mathrm{Im}\,\omega>0$, this change of variables is single-valued and well defined. Under this change the cut in the complex plane λ along the positive real axis from $|\zeta|^2$ to ∞, i.e. $\mathrm{Im}\,\lambda=0,\quad|\zeta|^2<\mathrm{Re}\,\lambda<\infty$, is mapped onto the whole real axis $\mathrm{Im}\,\omega=0,\quad -\infty<\mathrm{Re}\,\omega<+\infty$. The interval $\mathrm{Im}\,\lambda=0,\quad 0<\mathrm{Re}\,\lambda<|\zeta|^2$ on the real axis of λ is mapped onto an interval $\mathrm{Re}\,\omega=0,\quad 0<\mathrm{Im}\,\omega<|\zeta|$, on the *positive imaginary* axis of ω. As a function of ω the resolvent G is a meromorphic function in the upper half plane, $\mathrm{Im}\,\omega>0$, with simple poles on the interval $\mathrm{Re}\,\omega=0,\quad 0<\mathrm{Im}\,\omega<|\zeta|$, on the imaginary axis. The contour of integration in the complex plane of ω is a hyperbola going from $(e^{i3\pi/4})\infty$ through the point $\omega=\sqrt{|\zeta|^2-w}$ to $(e^{i\pi/4})\infty$. It can be deformed to a contour C that comes from $-\infty+i\varepsilon$, encircles the point $\omega=i|\zeta|$ in the clockwise direction and goes to $+\infty+i\varepsilon$, where ε is an infinitesimal positive parameter. The contour C does *not* cross the interval $\mathrm{Re}\,\omega=0,\quad 0<\mathrm{Im}\,\omega<|\zeta|$, on the imaginary axis and is *above* all the singularities of the resolvent G.

After such a transformation we obtain

$$\begin{aligned} U_0(t|x,y) &= (4\pi t)^{-m/2}\int\limits_{\mathbf{R}^{m-1}}\frac{d\zeta}{\pi^{(m-1)/2}}\exp\left\{-|\zeta|^2+i\zeta\cdot\frac{(\hat{x}-\hat{y})}{\sqrt{t}}\right\} \\ &\quad\times\int\limits_{C}\frac{d\omega}{\sqrt{\pi}}\,e^{-\omega^2}\,2(-i\omega)\,G\left(|\zeta|^2+\omega^2\Big|\zeta,\frac{r(x)}{\sqrt{t}},\frac{r(y)}{\sqrt{t}}\right). \end{aligned} \tag{6.47}$$

Substituting here G given in Eq. (6.45) and computing Gaussian integrals over ω and ζ we obtain the "free" part

$$U_{\infty(0)}(t|x,y)=(4\pi t)^{-m/2}\exp\left(-\frac{|x-y|^2}{4t}\right)I, \tag{6.48}$$

and the boundary part

$$\begin{aligned} U_{B(0)}(t|x,y) &= (4\pi t)^{-m/2}\Bigg\{\exp\left\{-\frac{|\hat{x}-\hat{y}|^2+[r(x)+r(y)]^2}{4t}\right\}(I-2\Pi) \\ &\quad+\Omega(t|x,y)\Bigg\}, \end{aligned} \tag{6.49}$$

where

$$\Omega(t|x,y) = -2\int\limits_{\mathbf{R}^{m-1}} \frac{d\zeta}{\pi^{(m-1)/2}} \exp\left\{-|\zeta|^2 + i\zeta\cdot\frac{(\hat{x}-\hat{y})}{\sqrt{t}}\right\} \times \int\limits_C \frac{d\omega}{\sqrt{\pi}} \exp\left\{-\omega^2 + i\omega\frac{[r(x)+r(y)]}{\sqrt{t}}\right\}\Gamma\cdot\zeta(\omega\, I + \Gamma\cdot\zeta)^{-1}. \quad (6.50)$$

Herefrom we obtain easily the diagonal value of the heat kernel:

$$U_{(0)}(t|x,x) = (4\pi t)^{-m/2}\left\{I + \exp\left(-\frac{r^2(x)}{t}\right)(I-2\Pi) + \Phi\left(\frac{r(x)}{\sqrt{t}}\right)\right\}, \quad (6.51)$$

where

$$\Phi(z) = -2\int\limits_{\mathbf{R}^{m-1}} \frac{d\zeta}{\pi^{(m-1)/2}} \int\limits_C \frac{d\omega}{\sqrt{\pi}}\, e^{-|\zeta|^2-\omega^2+2i\omega z}\,\Gamma\cdot\zeta(\omega\, I + \Gamma\cdot\zeta)^{-1}. \quad (6.52)$$

This function can be expressed further as[1]

$$\Phi(z) = -2e^{-z^2}I - 2\frac{\partial}{\partial z}\Psi(z), \quad (6.53)$$

where

$$\Psi(z) = \int\limits_{\mathbf{R}^{m-1}} \frac{d\zeta}{\pi^{(m-1)/2}} \int_0^\infty dp\, \exp\left\{-|\zeta|^2 - (p+z)^2 + 2ip\Gamma\cdot\zeta\right\}. \quad (6.54)$$

It is not difficult to show that, as $z\to\infty$, the functions Ψ and Φ are *exponentially small*:

$$\Psi(z) \sim \frac{1}{2z}e^{-z^2}\left[I - \frac{1}{2z^2}(I+\Gamma^2) + \mathrm{O}(z^{-4})\right], \quad (6.55)$$

$$\Phi(z) \sim \frac{1}{z^2}e^{-z^2}\left[-\Gamma^2 + \mathrm{O}(z^{-2})\right], \quad (6.56)$$

where $\Gamma^2 \equiv g_{ij}\Gamma^i\Gamma^j$. For $z=0$, by using the change $\zeta\to-\zeta$, we obtain

$$\Psi(0) = \frac{\sqrt{\pi}}{2}\int\limits_{\mathbf{R}^{m-1}} \frac{d\zeta}{\pi^{(m-1)/2}} \exp\left\{-|\zeta|^2 - (\Gamma\cdot\zeta)^2\right\}. \quad (6.57)$$

Note that this integral converges only when the strong ellipticity condition $|\zeta|^2 I + (\Gamma\cdot\zeta)^2 > 0$ is satisfied.

7 $A_{1/2}(F,B)$ Coefficient

Now we take the diagonal $U_{(0)}(t|x,x)$ given by (6.51) and integrate over the manifold M. Because the boundary part $U_{B(0)}$ is exponentially small as $r(x)\to\infty$ we can in fact integrate it only over a narrow strip near the boundary, when $0<r(x)<\delta$. The difference is asymptotically small as $t\to 0^+$. Doing the change of variables $z=r/\sqrt{t}$ we reduce the integration to $0<z<\delta/\sqrt{t}$. We see that as $t\to 0^+$ we can integrate over z from 0 to ∞. The error is asymptotically small as $t\to 0^+$ and does not contribute to the asymptotic expansion of the trace of the heat kernel.

Thus, we obtain

$$\begin{aligned}\mathrm{Tr}_{L^2}\exp(-tF) &= \int_M \mathrm{dvol}(x)\mathrm{tr}_V U_0(t|x,x)+O(t^{-m/2+1})\\ &= (4\pi t)^{-m/2}\left\{A_0+\sqrt{t}\,A_{1/2}(F,B)+O(t)\right\},\end{aligned} \tag{7.58}$$

where A_0 is given by (4.18) and

$$A_{1/2}(F,B)=\int\limits_{\partial M}\mathrm{dvol}(\hat{x})\mathrm{tr}_V b_{1/2}, \tag{7.59}$$

with

$$b_{1/2}=-\frac{\sqrt{\pi}}{2}(I+2\Pi)+2\Psi(0) \tag{7.60}$$

Now, using (6.53) and (6.57) and the fact that $\Psi(\infty)=0$ we get easily

$$b_{1/2}=-\frac{\sqrt{\pi}}{2}(I+2\Pi)+\sqrt{\pi}\int\limits_{\mathbf{R}^{m-1}}\frac{d\zeta}{\pi^{(m-1)/2}}\exp\left\{-|\zeta|^2-(\Gamma\cdot\zeta)^2\right\}. \tag{7.61}$$

Note again that this integral converges *only* when the strong ellipticity condition is satisfied, i.e. $|\zeta|^2I+(\Gamma\cdot\zeta)^2>0$.

Further calculations of general nature, without knowing the algebraic properties of the matrices Γ^j, seem to be impossible. One can, however, evaluate the integral in form of an expansion in the matrices Γ^i. The integral over ζ becomes Gaussian, which enables one to obtain

$$b_{1/2}=\frac{\sqrt{\pi}}{2}\left\{I-2\Pi+2\sum_{n\geq 1}(-1)^n\frac{(2n)!}{(n!)^2 2^{2n}}\hat{g}_{i_1 i_2}\cdots\hat{g}_{i_{2n-1}i_{2n}}\Gamma^{(i_1}\cdots\Gamma^{i_{2n})}\right\}. \tag{7.62}$$

Since our main result (7.61) is rather complicated, we now consider two particular cases of physical relevance.

I. The first non-trivial case is when the matrices Γ^i form an Abelian algebra, i.e.

$$[\Gamma^i,\Gamma^j]=0. \tag{7.63}$$

One can then easily compute the integral (7.61) explicitly and obtain

$$b_{1/2} = \frac{\sqrt{\pi}}{2}\left\{-I - 2\Pi + 2(I+\Gamma^2)^{-1/2}\right\}. \tag{7.64}$$

In the case $\Gamma = 0$ we recover the familiar result for mixed boundary conditions.[4, 6] In the case $\Pi = 0$, this coincides with the result of Ref. 8, where the authors considered the particular case of commuting Γ^i matrices (without noting this explicitly).

II. A very important case is when the operator Λ is a *natural* operator on the boundary. Since it is of first order it can be only the generalized Dirac operator. In this case the matrices Γ^j satisfy a Dirac-type condition

$$\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2\,\hat{g}^{ij}\frac{1}{(m-1)}\Gamma^2, \tag{7.65}$$

which leads to[1]

$$b_{1/2} = \frac{\sqrt{\pi}}{2}\left\{-I - 2\Pi + 2\left(I + \frac{1}{(m-1)}\Gamma^2\right)^{-(m-1)/2}\right\}. \tag{7.66}$$

Note that this *differs substantially* from the result of Ref. 8, and shows again that the result of Ref. 8 applies actually only to the completely Abelian case, when all matrices Γ^j commute. Note also that, in the most interesting applications (e.g. in quantum gravity), the matrices Γ^i do not commute.[10] The result (7.61), however, is valid in the most general case. A particular realization of the above situation is the "pure" Dirac case when $\Gamma^2 = -\kappa(m-1)(I-\Pi)$, where κ is a constant. In this case we have [1]

$$b_{1/2} = \frac{\sqrt{\pi}}{2}\left\{-\Pi + (I-\Pi)\left[2(1-\kappa)^{-(m-1)/2} - 1\right]\right\}. \tag{7.67}$$

Thus, a singularity is found at $\kappa = 1$. This happens because, for $\kappa = 1$, the strong ellipticity condition is violated (see also Ref. 9). Indeed, the strong ellipticity condition (3.11),

$$(\Gamma\cdot\zeta)^2 + |\zeta|^2 I = |\zeta|^2[\Pi + (1-\kappa)(I-\Pi)] > 0, \tag{7.68}$$

implies in this case $\kappa < 1$ (cf. Ref. 9). This is a general feature of the Gilkey–Smith boundary-value problem: the heat-kernel coefficients have singularities when the strong ellipticity condition is violated.[1]

8 Boundary Singularities in the Non-Elliptic Case

Let us now consider the special case when V is a spin-tensor bundle and the boundary operator Λ is a *natural* operator, i.e. the matrices Γ^j form a representation of Spin(m). In other words, the matrices Γ^j can be constructed only from natural objects like the metric, the normal, the Dirac matrices and the frame. Then the eigenvalues of the leading symbol of the operator Λ, i.e. the matrix $\Gamma\cdot\zeta$, depend only on $|\zeta|$, and hence are linear in $|\zeta|$:

$$\mathrm{spec}\,(\Gamma\cdot\zeta) = \Big\{0,\dots,0,\pm i\nu_{(k)}|\zeta|;\ \ d_{(k)}\Big\} \tag{8.69}$$

where $\nu_{(k)}$ are some *positive* constants and $d_{(k)}$ are the corresponding *multiplicities.* Here the index (k) labels all *non-zero* eigenvalues. It is put in round brackets to avoid any confusion with the boundary coordinate index j. Then the strong ellipticity condition (3.11) implies

$$0 < \nu_{(k)} < 1. \tag{8.70}$$

Let us compute the fibre trace of the heat-kernel diagonal. Taking the trace of (6.51) we get

$$\mathrm{tr}_V U_0(t|x,x) = (4\pi t)^{-m/2}\left\{c_0 + c_1\exp\left[-\frac{r(x)^2}{t}\right] + J\left(\frac{r}{\sqrt{t}}\right)\right\}, \tag{8.71}$$

where

$$c_0 \equiv \mathrm{tr}_V I, \qquad c_1 \equiv \mathrm{tr}_V(I-2\Pi), \tag{8.72}$$

$$\begin{aligned} J(z) &\equiv \mathrm{tr}_V\Phi(z) \\ &= -4\sum_{(k)} d_{(k)} \int\limits_{\mathbf{R}^{m-1}} \frac{d\zeta}{\pi^{(m-1)/2}} \int\limits_C \frac{d\omega}{\sqrt{\pi}}\, e^{-|\zeta|^2-\omega^2+2i\omega z}\, \frac{\nu_{(k)}^2|\zeta|^2}{\omega^2+\nu_{(k)}^2|\zeta|^2}. \end{aligned} \tag{8.73}$$

Remember that the contour C lies in the upper half plane: it comes from $-\infty+i\varepsilon$, encircles the point $\omega = i|\zeta|$ in the clockwise direction and goes to $+\infty+i\varepsilon$.

We want to compute the asymptotics of the parametrix as $r\to 0$ while t is being fixed. This corresponds to the limit $z\to 0^+$. The first two terms are well defined. The real problem is the asymptotics of the function J as $z\to 0^+$. The integral over ω is calculated by using the formula

$$\int\limits_C d\omega\, f(\omega) = -2\pi i\,\mathrm{Res}_{\omega=i\nu_{(k)}|\zeta|}\, f(\omega) + \int\limits_{-\infty}^{\infty} d\omega\, f(\omega). \tag{8.74}$$

The integrals over ζ can be reduced to Gaussian integrals by lifting the denominator in the exponent or by using spherical coordinates.

Note that in the integral over ω all poles of the integrand lie on the imaginary axis. Now, if the strong ellipticity condition (8.70) is satisfied, i.e. all $\nu_{(k)} < 1$, then they do not

reach the point $i|\zeta|$. This is very important. This simple fact leads then to convergence of the integral over ζ and *regularity* of the limit $z \to 0^+$. Therefore, the heat-kernel diagonal is integrable near the boundary, leading to the asymptotics obtained in the previous Section.

Let us instead suppose that the *strong ellipticity condition is violated* in that there is an eigenvalue $\nu = 1$ with a multiplicity d. As is shown in Ref. 1, in this case the same procedure leads to a *singularity* of the function J as $z \to 0^+$, i.e.

$$J(z) \sim 2d(m-1)\Gamma\left(\frac{m}{2}\right)\frac{1}{z^m}, \tag{8.75}$$

and hence to a singularity of the parametrix near the boundary when $r \to 0$, t being fixed. Moreover, this singularity is *not integrable*, which means that the L^2 trace of the heat kernel, $\mathrm{Tr}_{L^2} \exp(-tF)$, does not exist at all! This is also reflected in the fact that the heat-kernel coefficients $A_{k/2}$ become singular. In other words, the standard form (4.17) of the asymptotic expansion of the heat kernel is no longer valid.

The singularity at the point $z = 0$ results exactly from the pole at $\omega = i|\zeta|$. In the strongly elliptic case all poles lie on the positive imaginary line with $\mathrm{Im}\,\omega < i|\zeta|$, so that there is a *finite* gap between the pole located at the point with the largest value of the imaginary part and the point $i|\zeta|$.

In the heat-kernel diagonal there are three types of terms now. The first class of terms do not vanish exponentially when $r \to \infty$. Those are the interior terms. They give the familiar interior contribution when integrated over a compact manifold. The second class of terms are those which are exponentially small when $r \to \infty$ and when $t \to 0^+$. These are the boundary terms. When integrated over the manifold they produce the boundary terms in the standard heat-kernel asymptotics. In fact, these terms behave, as $t \to 0^+$, as distributions near the boundary, so that they give well defined non-vanishing contributions (in form of integrals over the boundary) when integrated with a function. In the non-elliptic case we have however obtained also a third term. This term has an *unusual non-integrable singularity* at the boundary as $r \to 0$ (on fixing t)

$$\mathrm{tr}_V U_0(t|x,x) \overset{r\to 0}{\sim} (4\pi)^{-m/2} 2d(m-1)\Gamma(m/2)\frac{1}{r^m}. \tag{8.76}$$

Such a singularity is *non-standard* in that: i) it does not depend on t and ii) it is *not integrable* over r near the boundary, as $r \to 0$. This is a direct consequence of the violation of strong ellipticity.

One can ask: what if the strong ellipticity condition (8.70) is violated "strongly", i.e. there are some eigenvalues that are *larger* than one, $\nu > 1$? Well, then it is not difficult to see that the integrals (8.73) defining the function J diverge for *any* z. Thus, in this case the parametrix itself, not only its functional trace, does not exist at all!

9 Ellipticity in Linearized Gauge Theories

In this section we are going to show how the Gilkey–Smith boundary-value problem can be formulated in a general gauge theory, following Ref. 1. A linearized gauge theory is defined by two vector bundles, V and G, such that $\dim V > \dim G$. V is the bundle of gauge fields $\varphi \in C^\infty(V, M)$, and G (usually a group) is the bundle of parameters of gauge transformations $\epsilon \in C^\infty(G, M)$. Both bundles V and G are equipped with some positive-definite metrics here denoted by E and γ, respectively, that are Hermitian: $E^\dagger = E, \gamma^\dagger = \gamma$, and with the corresponding natural L^2 scalar products $(,)_V$ and $(,)_G$.

The gauge transformations are described by a *first-order* differential operator $R : C^\infty(G, M) \to C^\infty(V, M)$. We restrict to the most important case when the second-order operator $L : C^\infty(G, M) \to C^\infty(G, M)$ defined by $L = \bar{R}R$, where $\bar{R} = \gamma^{-1}R^\dagger E$, is a *Laplace-type operator* with a non-degenerate leading symbol $\sigma_L(L; \xi) = |\xi|^2 I_G$. This means that $\operatorname{rank} \sigma_L(R) = \dim G$.

The dynamics of gauge fields $\varphi \in C^\infty(V, M)$ at the linearized level is described by a gauge-invariant and formally self-adjoint *second-order* differential operator $\Delta : C^\infty(V, M) \to C^\infty(V, M)$. It is gauge-invariant in the sense that its leading symbol is *degenerate* and satisfies the identities

$$\sigma_L(\Delta)\sigma_L(R) = \sigma_L(\bar{R})\sigma_L(\Delta) = 0. \tag{9.77}$$

We also assume that $\operatorname{Ker} \sigma_L(\Delta) = \{\sigma_L(R)\epsilon \mid \epsilon \in G\}$, and hence $\operatorname{rank} \sigma_L(\Delta) = \dim V - \dim G$.

Instead of the operator Δ we introduce another formally self-adjoint second-order operator $F : C^\infty(V, M) \to C^\infty(V, M)$ by

$$F \equiv \Delta + R\bar{R}. \tag{9.78}$$

Here we again restrict ourselves to the most important case when F is also a *Laplace-type* operator with a non-degenerate leading symbol $\sigma_L(F; \xi) = |\xi|^2 I_V$. Both restrictions made so far are satisfied in many interesting examples, like Yang-Mills and Einstein theories (for more details, see Ref. 1)

In quantum field theory one is interested in the one-loop effective action which is expressed in terms of the functional determinants of the operators F and L by

$$\Gamma^{(1)} = \frac{1}{2}\log\, \mathrm{Det}\, F - \log\, \mathrm{Det}\, L. \tag{9.79}$$

On manifolds with boundary one has to impose some boundary conditions to make these operators self-adjoint and elliptic. In gauge theories one tries to choose the boundary conditions in a gauge-invariant way. Interestingly, this requirement fixes completely the form of the boundary operators associated to the operators F and L, respectively.

Let us define restrictions of the leading symbols of the operators R and Δ to the boundary, i.e.

$$\Pi \equiv \sigma_L(\Delta; N)\Big|_{\partial M}, \qquad \nu \equiv \sigma_L(R; N)\Big|_{\partial M}, \qquad \mu \equiv \sigma_L(R; \zeta)\Big|_{\partial M}. \tag{9.80}$$

Since $L = \bar{R}R$ and $F = \Delta + R\bar{R}$ are Laplace-type operators, it follows that $\bar{\nu}\nu = I_G$ and $\Pi = I_V - \nu\bar{\nu}$. Therefore, Π is a self-adjoint projector orthogonal to ν, $\bar{\Pi} = \Pi$, $\Pi\nu = \bar{\nu}\Pi = 0$.

The requirement of gauge invariance of the boundary conditions determines in an almost unique way that the boundary conditions for the operator L should be of Dirichlet type,

$$\epsilon\Big|_{\partial M} = 0, \tag{9.81}$$

and the boundary conditions for the operator F should read

$$\Pi\varphi\Big|_{\partial M} = 0, \qquad \bar{R}\varphi\Big|_{\partial M} = 0. \tag{9.82}$$

Since the operator $\bar{R}$ in the boundary conditions (9.82) is a first-order operator, the set of boundary conditions (9.82) is equivalent to the general Gilkey–Smith scheme formulated in Sec. 2. Separating the normal derivative in the operator $\bar{R}$ and denoting by W_0 the restriction of the vector bundle V to the boundary, we find exactly the Gilkey–Smith boundary conditions (2.3) with the boundary operator B of the form (2.5) involving a first-order operator $\Lambda : C^\infty(W_0, \partial M) \to C^\infty(W_0, \partial M)$, the matrices Γ^j being of the form

$$\Gamma^j = -\nu\bar{\nu}\mu^j\bar{\nu}. \tag{9.83}$$

These matrices are anti-self-adjoint, $\bar{\Gamma}^i = -\Gamma^i$, and orthogonal to the projector Π, i.e. $\Pi\Gamma^i = \Gamma^i\Pi = 0$.

The condition of strong ellipticity then means that the matrix $|\zeta|I - i\Gamma\cdot\zeta = |\zeta|I + i\nu\bar{\nu}\mu\bar{\nu}$ should be positive-definite. The sufficient condition (3.11) of ellipticity now reads

$$|\zeta|^2 I + (\Gamma\cdot\zeta)^2 = |\zeta|^2\Pi + (I - \Pi)[|\zeta|^2 I - \mu\bar{\mu}](I - \Pi) > 0. \tag{9.84}$$

Since for non-vanishing ζ the part proportional to Π is positive-definite, the condition of strong ellipticity takes the form

$$(I - \Pi)[|\zeta|^2 I - \mu\bar{\mu}](I - \Pi) > 0. \tag{9.85}$$

Thus, the following theorem is found to hold: [1]

Theorem 2 *The boundary-value problem (F, B) with the boundary operator B determined by the boundary conditions (9.82) is gauge-invariant provided that the boundary operator associated to the operator L takes the Dirichlet form. Moreover, it is strongly elliptic with respect to the cone $\mathbf{C} - \mathbf{R}_+$ if and only if the matrix $[|\zeta|I + i\nu\bar{\nu}\mu\bar{\nu}]$ is positive-definite. A sufficient condition for that reads*

$$(I - \Pi)[|\zeta|^2 I - \mu\bar{\mu}](I - \Pi) > 0. \tag{9.86}$$

We study some explicit examples of gauge theories, including Yang–Mills model and Einstein quantum gravity, in Ref. 1 and in another contribution to this volume.[13]

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